

## D01DAF – NAG Fortran Library Routine Document

**Note.** Before using this routine, please read the Users' Note for your implementation to check the interpretation of bold italicised terms and other implementation-dependent details.

### 1 Purpose

D01DAF attempts to evaluate a double integral to a specified absolute accuracy by repeated applications of the method described by Patterson.

### 2 Specification

```

SUBROUTINE D01DAF(YA, YB, PHI1, PHI2, F, ABSACC, ANS, NPTS, IFAIL)
INTEGER          NPTS, IFAIL
real           YA, YB, PHI1, PHI2, F, ABSACC, ANS
EXTERNAL        PHI1, PHI2, F

```

### 3 Description

This routine attempts to evaluate a definite integral of the form

$$I = \int_a^b \int_{\phi_1(y)}^{\phi_2(y)} f(x, y) dx dy$$

where  $a$  and  $b$  are constants and  $\phi_1(y)$  and  $\phi_2(y)$  are functions of the variable  $y$ .

The integral is evaluated by expressing it as

$$I = \int_a^b F(y) dy, \quad \text{where } F(y) = \int_{\phi_1(y)}^{\phi_2(y)} f(x, y) dx.$$

Both the outer integral  $I$  and the inner integrals  $F(y)$  are evaluated by the method, described by Patterson [1] and [2], of the optimum addition of points to Gauss quadrature formulae.

This method uses a family of interlacing common point formulae. Beginning with the three-point Gauss rule, formulae using 7, 15, 31, 63, 127 and finally 255 points are derived. Each new formula contains all the pivots of the earlier formulae so that no function evaluations are wasted. Each integral is evaluated by applying these formulae successively until two results are obtained which differ by less than the specified absolute accuracy.

### 4 References

- [1] Patterson T N L (1969) The optimum addition of points to quadrature formulae, errata *Math. Comput.* **23** 892
- [2] Patterson T N L (1968) On some Gauss and Lobatto based integration formulae *Math. Comput.* **22** 877–881

### 5 Parameters

- 1: YA — **real** *Input*  
*On entry:* the lower limit of the integral,  $a$ .
- 2: YB — **real** *Input*  
*On entry:* the upper limit of the integral,  $b$ . It is not necessary that  $a < b$ .

- 3:** PHI1 — *real* FUNCTION, supplied by the user. *External Procedure*  
 PHI1 must return the lower limit of the inner integral for a given value of  $y$ .  
 Its specification is:

```

real FUNCTION PHI1(Y)
real          Y

1:  Y — real Input
    On entry: the value of  $y$  for which the lower limit must be evaluated.
  
```

PHI1 must be declared as EXTERNAL in the (sub)program from which D01DAF is called. Parameters denoted as *Input* must **not** be changed by this procedure.

- 4:** PHI2 — *real* FUNCTION, supplied by the user. *External Procedure*  
 PHI2 must return the upper limit of the inner integral for a given value of  $y$ .  
 Its specification is:

```

real FUNCTION PHI2(Y)
real          Y

1:  Y — real Input
    On entry: the value of  $y$  for which the upper limit must be evaluated.
  
```

PHI2 must be declared as EXTERNAL in the (sub)program from which D01DAF is called. Parameters denoted as *Input* must **not** be changed by this procedure.

- 5:** F — *real* FUNCTION, supplied by the user. *External Procedure*  
 F must return the value of the integrand  $f$  at a given point.  
 Its specification is:

```

real FUNCTION F(X, Y)
real          X, Y

1:  X — real Input
2:  Y — real Input
    On entry: the co-ordinates of the point  $(x, y)$  at which the integrand must be evaluated.
  
```

F must be declared as EXTERNAL in the (sub)program from which D01DAF is called. Parameters denoted as *Input* must **not** be changed by this procedure.

- 6:** ABSACC — *real* *Input*  
*On entry:* the absolute accuracy requested.
- 7:** ANS — *real* *Output*  
*On exit:* the estimate of the integral.
- 8:** NPTS — INTEGER *Output*  
*On exit:* the total number of function evaluations.

**9: IFAIL — INTEGER***Input/Output*

*On entry:* IFAIL must be set to 0,  $-1$  or 1. Users who are unfamiliar with this parameter should refer to Chapter P01 for details.

*On exit:* IFAIL = 0 unless the routine detects an error or gives a warning (see Section 6).

**For this routine**, because the values of output parameters may be useful even if IFAIL  $\neq$  0 on exit, users are recommended to set IFAIL to  $-1$  before entry. **It is then essential to test the value of IFAIL on exit.** To suppress the output of an error message when soft failure occurs, set IFAIL to 1.

## 6 Error Indicators and Warnings

Errors or warnings specified by the routine:

IFAIL = 1

This indicates that 255 points have been used in the outer integral and convergence has not been obtained. All the inner integrals have, however, converged. In this case ANS may still contain an approximate estimate of the integral.

IFAIL = 10 \* N

This indicates that the outer integral has converged but N inner integrals have failed to converge with the use of 255 points. In this case ANS may still contain an approximate estimate of the integral, but its reliability will decrease as N increases.

IFAIL = 10 \* N + 1

This indicates that both the outer integral and N of the inner integrals have not converged. ANS may still contain an approximate estimate of the integral, but its reliability will decrease as N increases.

## 7 Accuracy

The absolute accuracy is specified by the variable ABSACC. If, on exit, IFAIL = 0 then the result is most likely correct to this accuracy. Even if IFAIL is non-zero on exit, it is still possible that the calculated result could differ from the true value by less than the given accuracy.

## 8 Further Comments

The time taken by the routine depends upon the complexity of the integrand and the accuracy requested.

With Patterson's method accidental convergence may occasionally occur, when two estimates of an integral agree to within the requested accuracy, but both estimates differ considerably from the true result. This could occur in either the outer integral or in one or more of the inner integrals.

If it occurs in the outer integral then apparent convergence is likely to be obtained with considerably fewer integrand evaluations than may be expected. If it occurs in an inner integral, the incorrect value could make the function  $F(y)$  appear to be badly behaved, in which case a very large number of pivots may be needed for the overall evaluation of the integral. Thus both unexpectedly small and unexpectedly large numbers of integrand evaluations should be considered as indicating possible trouble. If accidental convergence is suspected, the integral may be recomputed, requesting better accuracy; if the new request is more stringent than the degree of accidental agreement (which is of course unknown), improved results should be obtained. This is only possible when the accidental agreement is not better than machine accuracy. It should be noted that the routine requests the same accuracy for the inner integrals as for the outer integral. In practice it has been found that in the vast majority of cases this has proved to be adequate for the overall result of the double integral to be accurate to within the specified value.

The routine is not well-suited to non-smooth integrands, i.e., integrands having some kind of analytic discontinuity (such as a discontinuous or infinite partial derivative of some low order) in, on the boundary

of, or near, the region of integration. **Warning:** such singularities may be induced by incautiously presenting an apparently smooth interval over the positive quadrant of the unit circle,  $R$

$$I = \int_R (x + y) dx dy.$$

This may be presented to D01DAF as

$$I = \int_0^1 dy \int_0^{\sqrt{1-y^2}} (x + y) dx = \int_0^1 \left( \frac{1}{2}(1 - y^2) + y\sqrt{1 - y^2} \right) dy$$

but here the outer integral has an induced square-root singularity stemming from the way the region has been presented to D01DAF. This situation should be avoided by re-casting the problem. For the example given, the use of polar co-ordinates would avoid the difficulty:

$$I = \int_0^1 dr \int_0^{\frac{\pi}{2}} r^2(\cos v + \sin v)dv.$$

## 9 Example

The following program evaluates the integral discussed in Section 8, presenting it to D01DAF first as

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x + y) dx dy$$

and then as

$$\int_0^1 \int_0^{\frac{\pi}{2}} r^2(\cos v + \sin v)dv dr.$$

Note the difference in the number of function evaluations.

### 9.1 Program Text

**Note.** The listing of the example program presented below uses bold italicised terms to denote precision-dependent details. Please read the Users' Note for your implementation to check the interpretation of these terms. As explained in the Essential Introduction to this manual, the results produced may not be identical for all implementations.

```

*      D01DAF Example Program Text
*      Mark 14 Revised.  NAG Copyright 1989.
*      .. Parameters ..
      INTEGER          NOUT
      PARAMETER       (NOUT=6)
*      .. Local Scalars ..
      real            ABSACC, ANS, YA, YB
      INTEGER          IFAIL, NPTS
*      .. External Functions ..
      real            FA, FB, P1, P2A, P2B
      EXTERNAL         FA, FB, P1, P2A, P2B
*      .. External Subroutines ..
      EXTERNAL        D01DAF
*      .. Executable Statements ..
      WRITE (NOUT,*) 'D01DAF Example Program Results'
      YA = 0.0e0
      YB = 1.0e0
      ABSACC = 1.0e-6
      WRITE (NOUT,*)
      WRITE (NOUT,*) 'First formulation'
      IFAIL = 1
*
*      CALL D01DAF(YA,YB,P1,P2A,FA,ABSACC,ANS,NPTS,IFAIL)
*

```

```

WRITE (NOUT,99999) 'Integral =', ANS
WRITE (NOUT,99998) 'Number of function evaluations =', NPTS
IF (IFAIL.GT.0) WRITE (NOUT,99997) 'IFAIL = ', IFAIL
WRITE (NOUT,*)
WRITE (NOUT,*) 'Second formulation'
IFAIL = 1
*
CALL D01DAF(YA,YB,P1,P2B,FB,ABSACC,ANS,NPTS,IFAIL)
*
WRITE (NOUT,99999) 'Integral =', ANS
WRITE (NOUT,99998) 'Number of function evaluations =', NPTS
IF (IFAIL.GT.0) WRITE (NOUT,99997) 'IFAIL = ', IFAIL
STOP
*
99999 FORMAT (1X,A,F9.4)
99998 FORMAT (1X,A,I5)
99997 FORMAT (1X,A,I2)
END
*
real FUNCTION P1(Y)
*
.. Scalar Arguments ..
real          Y
*
.. Executable Statements ..
P1 = 0.0e0
RETURN
END
*
real FUNCTION P2A(Y)
*
.. Scalar Arguments ..
real          Y
*
.. Intrinsic Functions ..
INTRINSIC      SQR
*
.. Executable Statements ..
P2A = SQR(1.0e0-Y*Y)
RETURN
END
*
real FUNCTION FA(X,Y)
*
.. Scalar Arguments ..
real          X, Y
*
.. Executable Statements ..
FA = X + Y
RETURN
END
*
real FUNCTION P2B(Y)
*
.. Scalar Arguments ..
real          Y
*
.. External Functions ..
real          X01AAF
EXTERNAL      X01AAF
*
.. Executable Statements ..
P2B = 0.5e0*X01AAF(0.0e0)
RETURN
END
*
real FUNCTION FB(X,Y)

```

```
*    .. Scalar Arguments ..  
  real          X, Y  
*    .. Intrinsic Functions ..  
  INTRINSIC      COS, SIN  
*    .. Executable Statements ..  
  FB = Y*Y*(COS(X)+SIN(X))  
  RETURN  
  END
```

## 9.2 Program Data

None.

## 9.3 Program Results

D01DAF Example Program Results

First formulation

Integral = 0.6667

Number of function evaluations = 189

Second formulation

Integral = 0.6667

Number of function evaluations = 89

---