NAG Fortran Library Routine Document

D03NDF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of ***bold italicised*** terms and other implementation-dependent details.

1 Purpose

D03NDF computes an analytic solution to the Black-Scholes equation for a certain set of option types.

2 Specification

```
 SUBROUTINE D03NDF(KOPT, X, S, T, TMAT, TDPAR, R, Q, SIGMA, F, THETA,
1                  DELTA, GAMMA, LAMBDA, RHO, IFAIL)
 INTEGER           KOPT, IFAIL
 real              X, S, T, TMAT, R(*), Q(*), SIGMA(*), F, THETA, DELTA,
1                  GAMMA, LAMBDA, RHO
 LOGICAL           TDPAR(3)
```

3 Description

D03NDF computes an analytic solution to the Black-Scholes equation (see Hull (1989) and Wilmott *et al.* (1995))

$$\frac{\partial f}{\partial t} + (r - q)S\frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2} = rf \tag{1}$$

$$S_{\min} < S < S_{\max}, \quad t_{\min} < t < t_{\max}, \tag{2}$$

for the value f of a European put or call option, or an American call option with zero dividend q. In equation (1) of the document for D03NDF t is time, S is the stock price, X is the exercise price, r is the risk free interest rate, q is the continuous dividend, and σ is the stock volatility. The parameters r, q and σ may be either constant, or functions of time. In the latter case their average instantaneous values over the remaining life of the option should be provided to D03NDF. An auxiliary routine D03NEF is available to compute such averages from values at a set of discrete times.

D03NDF also returns values of the Greeks

$$\Theta = \frac{\partial f}{\partial t}, \quad \Delta = \frac{\partial f}{\partial x}, \quad \Gamma = \frac{\partial^2 f}{\partial x^2}, \quad \Lambda = \frac{\partial f}{\partial \sigma}, \quad \rho = \frac{\partial f}{\partial r}.$$

Further details of the analytic solution returned are given in Section 8.1.

4 References

Hull J (1989) *Options, Futures and Other Derivative Securities* Prentice-Hall

Wilmott P, Howison S and Dewynne J (1995) *The Mathematics of Financial Derivatives* Cambridge University Press

5 Parameters

1: KOPT – INTEGER *Input*

On entry: specifies the kind of option to be valued:

KOPT = 1

European call.

KOPT = 2

American call.

KOPT = 3

European put.

Constraints:

$1 \le \text{KOPT} \le 3$,
$\text{KOPT} \ne 2$ when $q \ne 0$.

2: X – ***real*** *Input*

On entry: the exercise price X.

Constraint: $\text{X} \ge 0.0$.

3: S – ***real*** *Input*

On entry: the stock price at which the option value and the Greeks should be evaluated.

Constraint: $\text{S} \ge 0.0$.

4: T – ***real*** *Input*

On entry: the time at which the option value and the Greeks should be evaluated.

Constraint: $\text{T} \ge 0.0$.

5: TMAT – ***real*** *Input*

On entry: the maturity time of the option.

Constraint: $\text{TMAT} \ge \text{T}$.

6: TDPAR(3) – LOGICAL array *Input*

On entry: specifies whether or not various parameters are time-dependent. More precisely, r is time-dependent if TDPAR(1) = .TRUE. and constant otherwise. Similarly, TDPAR(2) specifies whether q is time-dependent and TDPAR(3) specifies whether σ is time-dependent.

7: R($*$) – ***real*** array *Input*

Note: the dimension of the array R must be at least 3 when TDPAR(1) = .TRUE., and at least 1 otherwise.

On entry: if TDPAR(1) = .FALSE. then R(1) must contain the constant value of r. The remaining elements need not be set. If TDPAR(1) = .TRUE. then R(1) must contain the value of r at time T and R(2) must contain its average instantaneous value over the remaining life of the option:

$$\hat{r} = \int_{\text{T}}^{\text{TMAT}} r(\zeta)\, d\zeta.$$

The auxiliary routine D03NEF may be used to construct R from a set of values of r at discrete times.

8: Q($*$) – ***real*** array *Input*

Note: the dimension of the array Q must be at least 3 when TDPAR(2) = .TRUE., and at least 1 otherwise.

On entry: if TDPAR(2) = .FALSE. then Q(1) must contain the constant value of q. The remaining elements need not be set. If TDPAR(2) = .TRUE. then Q(1) must contain the constant value of q and Q(2) must contain its average instantaneous value over the remaining life of the option:

$$\hat{q} = \int_{\mathrm{T}}^{\mathrm{TMAT}} q(\zeta)\ d\zeta.$$

The auxiliary routine D03NEF may be used to construct Q from a set of values of q at discrete times.

9: SIGMA(*) – ***real*** array *Input*

Note: the dimension of the array SIGMA must be at least 3 when TDPAR(3) = .TRUE., and at least 1 otherwise.

On entry: if TDPAR(3) = .FALSE. then SIGMA(1) must contain the constant value of σ. The remaining elements need not be set. If TDPAR(3) = .TRUE. then SIGMA(1) must contain the value of σ at time T, SIGMA(2) the average instantaneous value $\hat{\sigma}$, and SIGMA(3) the second-order average $\bar{\sigma}$, where:

$$\hat{\sigma} = \int_{\mathrm{T}}^{\mathrm{TMAT}} \sigma(\zeta)\ d\zeta,$$

$$\bar{\sigma} = \left(\int_{\mathrm{T}}^{\mathrm{TMAT}} \sigma^2(\zeta)\ d\zeta \right)^{1/2}.$$

The auxiliary routine D03NEF may be used to compute SIGMA from a set of values at discrete times.

Constraints:

SIGMA(1) > 0.0 when if TDPAR(3) = .FALSE.,
SIGMA(i) > 0.0, for $i = 1, 2, 3$, when TDPAR(3) = .TRUE..

10: F – ***real*** *Output*

On exit: the value f of the option at the stock price S and time T.

11: THETA – ***real*** *Output*
12: DELTA – ***real*** *Output*
13: GAMMA – ***real*** *Output*
14: LAMBDA – ***real*** *Output*
15: RHO – ***real*** *Output*

On exit: the values of various Greeks at the stock price S and time T, as follows:

$$\mathrm{THETA} = \Theta = \frac{\partial f}{\partial t}, \quad \mathrm{DELTA} = \Delta = \frac{\partial f}{\partial \mathrm{S}}, \quad \mathrm{GAMMA} = \Gamma = \frac{\partial^2 f}{\partial \mathrm{S}^2},$$

$$\mathrm{LAMBDA} = \Lambda = \frac{\partial f}{\partial \sigma}, \quad \mathrm{RHO} = \rho = \frac{\partial f}{\partial r}.$$

16: IFAIL – INTEGER *Input/Output*

On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to Chapter P01 for details.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, for users not familiar with this parameter the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

6 Error Indicators and Warnings

If on entry IFAIL = 0 or −1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL = 1

On entry, KOPT < 1,
or KOPT > 3,
or KOPT = 2 when $q \neq 0$,
or X < 0.0,
or S < 0.0,
or T < 0.0,
or TMAT < T,
or SIGMA(1) ≤ 0.0, with TDPAR(3) = .FALSE.,
or SIGMA(i) ≤ 0.0, with TDPAR(3) = .TRUE., for some $i = 1$, 2 or 3.

7 Accuracy

Given accurate values of R, Q and SIGMA no further approximations are made in the evaluation of the Black-Scholes analytic formulae, and the results should therefore be within machine accuracy. The values of R, Q and SIGMA returned from D03NEF are exact for polynomials of degree up to 3.

8 Further Comments

8.1 Algorithmic Details

The Black-Scholes analytic formulae are used to compute the solution. For a European call option these are as follows:

$$f = Se^{-\hat{q}(T-t)}N(d_1) - Xe^{-\hat{r}(T-t)}N(d_2)$$

where

$$d_1 = \frac{\log(S/X) + (\hat{r} - \hat{q} + \bar{\sigma}^2/2)(T-t)}{\bar{\sigma}\sqrt{T-t}},$$

$$d_2 = \frac{\log(\mathrm{S/X}) + (\hat{r} - \hat{q} - \bar{\sigma}^2/2)(T-t)}{\bar{\sigma}\sqrt{T-t}} = d_1 - \bar{\sigma}\sqrt{T-t},$$

$N(x)$ is the cumulative Normal distribution function and $N'(x)$ is its derivative

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\zeta^2/2}\, d\zeta,$$

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

The functions $\hat{q}$, $\hat{r}$, $\hat{\sigma}$ and $\bar{\sigma}$ are average values of q, r and σ over the time to maturity:

$$\hat{q} = \frac{1}{T-t}\int_t^T q(\zeta)\, d\zeta,$$

$$\hat{r} = \frac{1}{T-t}\int_t^T r(\zeta)\, d\zeta,$$

$$\hat{\sigma} = \frac{1}{T-t}\int_t^T \sigma(\zeta)\, d\zeta,$$

$$\bar{\sigma} = \left(\frac{1}{T-t} \int_t^T \sigma^2(\zeta)\ d\zeta \right)^{1/2}.$$

The Greeks are then calculated as follows:

$$\Delta = \frac{\partial f}{\partial S} = e^{-\hat{q}(T-t)} N(d_1) + \frac{S e^{-\hat{q}(T-t)} N'(d_1) - X e^{-\hat{r}(T-t)} N'(d_2)}{\bar{\sigma} S \sqrt{T-t}},$$

$$\Gamma = \frac{\partial^2 f}{\partial S^2} = \frac{S e^{-\hat{q}(T-t)} N'(d_1) + X e^{-\hat{r}(T-t)} N'(d_2)}{\bar{\sigma} S^2 \sqrt{T-t}} + \frac{S e^{-\hat{q}(T-t)} N'(d_1) - X e^{-\hat{r}(T-t)} N'(d_2)}{\bar{\sigma}^2 S^2 (T-t)},$$

$$\Theta = \frac{\partial f}{\partial t} = rf + (q-r)S\Delta - \frac{\sigma^2 S^2}{2}\Gamma,$$

$$\Lambda = \frac{\partial f}{\partial \sigma} = \left(\frac{X d_1 e^{-\hat{r}(T-t)} N'(d_2) - S d_2 e^{-\hat{q}(T-t)} N'(d_1)}{\bar{\sigma}^2} \right) \hat{\sigma},$$

$$\rho = \frac{\partial f}{\partial r} = X(T-t) e^{-\hat{r}(T-t)} N(d_2) + \frac{(S e^{-\hat{q}(T-t)} N'(d_1) - X e^{-\hat{r}(T-t)} N'(d_2))\sqrt{T-t}}{\bar{\sigma}}.$$

Note: that Θ is obtained from substitution of other Greeks in the Black-Scholes partial differential equation, rather than differentiation of f. The values of q, r and σ appearing in its definition are the instantaneous values, not the averages. Note also that both the first-order average $\hat{\sigma}$ and the second-order average $\bar{\sigma}$ appear in the expression for Λ. This results from the fact that Λ is the derivative of f with respect to σ, not $\hat{\sigma}$.

For a European put option the equivalent equations are:

$$f = X e^{-\hat{r}(T-t)} N(-d_2) - S e^{-\hat{q}(T-t)} N(-d_1),$$

$$\Delta = \frac{\partial f}{\partial S} = -e^{-\hat{q}(T-t)} N(-d_1) + \frac{S e^{-\hat{q}(T-t)} N'(-d_1) - X e^{-\hat{r}(T-t)} N'(-d_2)}{\bar{\sigma} S \sqrt{T-t}},$$

$$\Gamma = \frac{\partial^2 f}{\partial S^2} = \frac{X e^{-\hat{r}(T-t)} N'(-d_2) + S e^{-\hat{q}(T-t)} N'(-d_1)}{\bar{\sigma} S^2 \sqrt{T-t}} + \frac{X e^{-\hat{r}(T-t)} N''(-d_2) - S e^{-\hat{q}(T-t)} N''(-d_1)}{\bar{\sigma}^2 S^2 (T-t)},$$

$$\Theta = \frac{\partial f}{\partial t} = rf + (q-r)S\Delta - \frac{\sigma^2 S^2}{2}\Gamma,$$

$$\Lambda = \frac{\partial f}{\partial \sigma} = \left(\frac{X d_1 e^{-\hat{r}(T-t)} N'(-d_2) - S d_2 e^{-\hat{q}(T-t)} N'(-d_1)}{\bar{\sigma}^2} \right) \hat{\sigma},$$

$$\rho = \frac{\partial f}{\partial r} = -X(T-t) e^{-\hat{r}(T-t)} N(-d_2) + \frac{(S e^{-\hat{q}(T-t)} N'(-d_1) - X e^{-\hat{r}(T-t)} N'(-d_2))\sqrt{T-t}}{\hat{\sigma}}.$$

The analytic solution for an American call option with $q = 0$ is identical to that for a European call, since early exercise is never optimal in this case. For all other cases no analytic solution is known.

9 Example

This example solves the Black-Scholes equation for valuation of a 5-month American call option on a non-dividend-paying stock with an exercise price of \$50. The risk-free interest rate is 10% per annum, and the stock volatility is 40% per annum.

The option is valued at a range of times and stock prices.

9.1 Program Text

Note: the listing of the example program presented below uses ***bold italicised*** terms to denote precision-dependent details. Please read the Users' Note for your implementation to check the interpretation of these terms. As explained in the Essential Introduction to this manual, the results produced may not be identical for all implementations.

```
*     D03NDF Example Program Text
*     Mark 20 Release. NAG Copyright 2001.
*     .. Parameters ..
      INTEGER          NIN, NOUT
      PARAMETER        (NIN=5,NOUT=6)
      INTEGER          NSMAX, NTMAX
      PARAMETER        (NSMAX=100,NTMAX=100)
*     .. Local Scalars ..
      real             DS, DT, TMAT, X
      INTEGER          I, IFAIL, IGREEK, J, KOPT, NS, NT
*     .. Local Arrays ..
      real             DELTA(NSMAX,NTMAX), F(NSMAX,NTMAX),
     +                 GAMMA(NSMAX,NTMAX), LAMBDA(NSMAX,NTMAX), Q(3),
     +                 R(3), RHO(NSMAX,NTMAX), S(NSMAX), SIGMA(3),
     +                 T(NTMAX), THETA(NSMAX,NTMAX)
      LOGICAL          GPRNT(5), TDPAR(3)
      CHARACTER*6      GNAME(5)
*     .. External Subroutines ..
      EXTERNAL         D03NDF
*     .. Intrinsic Functions ..
      INTRINSIC        real
*     .. Data statements ..
      DATA             GNAME/'Theta ', 'Delta ', 'Gamma ', 'Lambda',
     +                 'Rho   '/
      DATA             GPRNT/5*.TRUE./
*     .. Executable Statements ..
      WRITE (NOUT,*) 'D03NDF Example Program Results'
      WRITE (NOUT,*)
*
*     Skip heading in data file
*
      READ (NIN,*)
*
*     Read problem parameters
*
      READ (NIN,*) KOPT
      READ (NIN,*) X
      READ (NIN,*) TMAT
      READ (NIN,*) R(1)
      READ (NIN,*) Q(1)
      READ (NIN,*) SIGMA(1)
      READ (NIN,*) NS, NT
      READ (NIN,*) S(1), S(NS)
      READ (NIN,*) T(1), T(NT)
*
      TDPAR(1) = .FALSE.
      TDPAR(2) = .FALSE.
      TDPAR(3) = .FALSE.

      IF (NS.LT.2 .OR. NS.GT.NSMAX) THEN
         WRITE (NOUT,*) 'NS invalid.'
      ELSE IF (NT.LT.2 .OR. NT.GT.NTMAX) THEN
         WRITE (NOUT,*) 'NT invalid.'
      ELSE
*
         DS = (S(NS)-S(1))/real(NS-1)
         DT = (T(NT)-T(1))/real(NT-1)
*
*         Loop over times
*
         DO 40 J = 1, NT
            T(J) = T(1) + (J-1)*DT
*
*            Loop over stock prices
*
```

```
            DO 20 I = 1, NS
               S(I) = S(1) + (I-1)*DS
*
*                 Call Black-Scholes solver
*
               IFAIL = 0
               CALL D03NDF(KOPT,X,S(I),T(J),TMAT,TDPAR,R,Q,SIGMA,F(I,J),
     +                     THETA(I,J),DELTA(I,J),GAMMA(I,J),LAMBDA(I,J),
     +                     RHO(I,J),IFAIL)
*
   20       CONTINUE
   40    CONTINUE
*
*        Output option values.
*
         WRITE (NOUT,*)
         WRITE (NOUT,*) 'Option Values'
         WRITE (NOUT,*) '-------------'
         WRITE (NOUT,*) ' Stock Price  |   Time to Maturity (months)'
         WRITE (NOUT,99999) '|', (12*(TMAT-T(J)),J=1,NT)
         WRITE (NOUT,'(11A)') '-----------------',
     +     ('------------',J=1,NT)
         DO 60 I = 1, NS
            WRITE (NOUT,99998) S(I), '|', (F(I,J),J=1,NT)
   60    CONTINUE
*
         DO 100 IGREEK = 1, 5
*
            IF (GPRNT(IGREEK)) THEN
               WRITE (NOUT,*)
               WRITE (NOUT,*) GNAME(IGREEK)
               WRITE (NOUT,*) '------'
               WRITE (NOUT,*)
     +           ' Stock Price  |   Time to Maturity (months)'
               WRITE (NOUT,99999) '|', (12*(TMAT-T(J)),J=1,NT)
               WRITE (NOUT,'(11A)') '-----------------',
     +           ('------------',J=1,NT)
               DO 80 I = 1, NS
                  IF (IGREEK.EQ.1) THEN
                     WRITE (NOUT,99998) S(I), '|', (THETA(I,J),J=1,NT)
                  ELSE IF (IGREEK.EQ.2) THEN
                     WRITE (NOUT,99998) S(I), '|', (DELTA(I,J),J=1,NT)
                  ELSE IF (IGREEK.EQ.3) THEN
                     WRITE (NOUT,99998) S(I), '|', (GAMMA(I,J),J=1,NT)
                  ELSE IF (IGREEK.EQ.4) THEN
                     WRITE (NOUT,99998) S(I), '|', (LAMBDA(I,J),J=1,NT)
                  ELSE IF (IGREEK.EQ.5) THEN
                     WRITE (NOUT,99998) S(I), '|', (RHO(I,J),J=1,NT)
                  END IF
   80          CONTINUE
            END IF
*
  100    CONTINUE
*
      END IF
*
      STOP
*
99999 FORMAT (15X,A,1X,12(1P,e12.4))
99998 FORMAT (1P,e12.4,3X,A,1X,12(1P,e12.4))
      END
```

9.2 Program Data

```
D03NDF Example Program Data
  2                     KOPT
  50.                   X
  0.4166667             TMAT
  0.1                   R(1)
  0.0                   Q(1)
```

```
 0.4                 SIGMA(1)
 21  4               NS, NT
 0.0  100.           S(1), S(NS)
 0.0  0.125          T(1), T(NT)
```

9.3 Program Results

```
D03NDF Example Program Results


Option Values
-------------
 Stock Price  |   Time to Maturity (months)
              |   5.0000E+00  4.5000E+00  4.0000E+00  3.5000E+00
-----------------------------------------------------------------
 0.0000E+00   |   0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00
 5.0000E+00   |   4.4491E-19  4.5989E-21  1.5461E-23  1.0478E-26
 1.0000E+01   |   5.5566E-10  5.5129E-11  3.1298E-12  8.0281E-14
 1.5000E+01   |   4.7337E-06  1.2187E-06  2.2774E-07  2.7003E-08
 2.0000E+01   |   7.2236E-04  3.1054E-04  1.1005E-04  2.9678E-05
 2.5000E+01   |   1.6557E-02  9.6610E-03  5.0099E-03  2.2012E-03
 3.0000E+01   |   1.3307E-01  9.4037E-02  6.1869E-02  3.6848E-02
 3.5000E+01   |   5.6631E-01  4.5257E-01  3.4667E-01  2.5053E-01
 4.0000E+01   |   1.6004E+00  1.3850E+00  1.1699E+00  9.5640E-01
 4.5000E+01   |   3.4384E+00  3.1328E+00  2.8168E+00  2.4891E+00
 5.0000E+01   |   6.1165E+00  5.7600E+00  5.3874E+00  4.9960E+00
 5.5000E+01   |   9.5300E+00  9.1645E+00  8.7846E+00  8.3882E+00
 6.0000E+01   |   1.3509E+01  1.3163E+01  1.2808E+01  1.2445E+01
 6.5000E+01   |   1.7883E+01  1.7568E+01  1.7251E+01  1.6932E+01
 7.0000E+01   |   2.2513E+01  2.2230E+01  2.1949E+01  2.1671E+01
 7.5000E+01   |   2.7301E+01  2.7045E+01  2.6792E+01  2.6544E+01
 8.0000E+01   |   3.2182E+01  3.1946E+01  3.1713E+01  3.1485E+01
 8.5000E+01   |   3.7117E+01  3.6894E+01  3.6674E+01  3.6458E+01
 9.0000E+01   |   4.2081E+01  4.1868E+01  4.1656E+01  4.1446E+01
 9.5000E+01   |   4.7062E+01  4.6854E+01  4.6647E+01  4.6441E+01
 1.0000E+02   |   5.2052E+01  5.1847E+01  5.1643E+01  5.1439E+01

Theta
------
 Stock Price  |   Time to Maturity (months)
              |   5.0000E+00  4.5000E+00  4.0000E+00  3.5000E+00
-----------------------------------------------------------------
 0.0000E+00   |   0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00
 5.0000E+00   |  -4.4017E-17 -5.5977E-19 -2.3735E-21 -2.0936E-24
 1.0000E+01   |  -2.7827E-08 -3.3857E-09 -2.4163E-10 -8.0398E-12
 1.5000E+01   |  -1.3953E-04 -4.3864E-05 -1.0258E-05 -1.5706E-06
 2.0000E+01   |  -1.3287E-02 -6.9342E-03 -3.0567E-03 -1.0576E-03
 2.5000E+01   |  -1.9512E-01 -1.3714E-01 -8.7730E-02 -4.9018E-02
 3.0000E+01   |  -1.0161E+00 -8.5596E-01 -6.8695E-01 -5.1395E-01
 3.5000E+01   |  -2.8112E+00 -2.6426E+00 -2.4328E+00 -2.1723E+00
 4.0000E+01   |  -5.1662E+00 -5.1709E+00 -5.1500E+00 -5.0892E+00
 4.5000E+01   |  -7.2196E+00 -7.4540E+00 -7.7180E+00 -8.0183E+00
 5.0000E+01   |  -8.3848E+00 -8.7388E+00 -9.1543E+00 -9.6525E+00
 5.5000E+01   |  -8.6152E+00 -8.9372E+00 -9.3056E+00 -9.7329E+00
 6.0000E+01   |  -8.2058E+00 -8.4077E+00 -8.6186E+00 -8.8343E+00
 6.5000E+01   |  -7.5116E+00 -7.5845E+00 -7.6368E+00 -7.6553E+00
 7.0000E+01   |  -6.7905E+00 -6.7711E+00 -6.7202E+00 -6.6262E+00
 7.5000E+01   |  -6.1758E+00 -6.1099E+00 -6.0160E+00 -5.8893E+00
 8.0000E+01   |  -5.7084E+00 -5.6310E+00 -5.5359E+00 -5.4234E+00
 8.5000E+01   |  -5.3786E+00 -5.3103E+00 -5.2340E+00 -5.1533E+00
 9.0000E+01   |  -5.1582E+00 -5.1071E+00 -5.0551E+00 -5.0062E+00
 9.5000E+01   |  -5.0165E+00 -4.9835E+00 -4.9536E+00 -4.9298E+00
 1.0000E+02   |  -4.9281E+00 -4.9107E+00 -4.8979E+00 -4.8916E+00

Delta
------
 Stock Price  |   Time to Maturity (months)
              |   5.0000E+00  4.5000E+00  4.0000E+00  3.5000E+00
-----------------------------------------------------------------
 0.0000E+00   |   0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00
```

```
  5.0000E+00   |   3.1381E-18  3.5969E-20  1.3576E-22  1.0494E-25
  1.0000E+01   |   1.4005E-09  1.5376E-10  9.7805E-12  2.8553E-13
  1.5000E+01   |   6.1418E-06  1.7452E-06  3.6436E-07  4.9030E-08
  2.0000E+01   |   5.6040E-04  2.6494E-04  1.0451E-04  3.1863E-05
  2.5000E+01   |   8.3312E-03  5.3217E-03  3.0570E-03  1.5104E-03
  3.0000E+01   |   4.5711E-02  3.5158E-02  2.5461E-02  1.6934E-02
  3.5000E+01   |   1.3765E-01  1.1889E-01  9.9459E-02  7.9557E-02
  4.0000E+01   |   2.8307E-01  2.6258E-01  2.3996E-01  2.1479E-01
  4.5000E+01   |   4.5320E-01  4.3858E-01  4.2214E-01  4.0335E-01
  5.0000E+01   |   6.1427E-01  6.0856E-01  6.0249E-01  5.9601E-01
  5.5000E+01   |   7.4525E-01  7.4687E-01  7.4937E-01  7.5308E-01
  6.0000E+01   |   8.4052E-01  8.4611E-01  8.5298E-01  8.6148E-01
  6.5000E+01   |   9.0433E-01  9.1096E-01  9.1862E-01  9.2752E-01
  7.0000E+01   |   9.4449E-01  9.5045E-01  9.5699E-01  9.6412E-01
  7.5000E+01   |   9.6862E-01  9.7325E-01  9.7808E-01  9.8300E-01
  8.0000E+01   |   9.8260E-01  9.8589E-01  9.8913E-01  9.9221E-01
  8.5000E+01   |   9.9050E-01  9.9269E-01  9.9473E-01  9.9653E-01
  9.0000E+01   |   9.9487E-01  9.9627E-01  9.9748E-01  9.9848E-01
  9.5000E+01   |   9.9725E-01  9.9811E-01  9.9881E-01  9.9935E-01
  1.0000E+02   |   9.9854E-01  9.9905E-01  9.9945E-01  9.9972E-01

 Gamma
 ------
  Stock Price  |   Time to Maturity (months)
               |   5.0000E+00  4.5000E+00  4.0000E+00  3.5000E+00
-----------------------------------------------------------------
  0.0000E+00   |   0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00
  5.0000E+00   |   2.1246E-17  2.7112E-19  1.1536E-21  1.0211E-24
  1.0000E+01   |   3.3102E-09  4.0468E-10  2.9020E-11  9.7029E-13
  1.5000E+01   |   7.2660E-06  2.2982E-06  5.4080E-07  8.3319E-08
  2.0000E+01   |   3.8245E-04  2.0111E-04  8.9333E-05  3.1153E-05
  2.5000E+01   |   3.5190E-03  2.4960E-03  1.6118E-03  9.0924E-04
  3.0000E+01   |   1.2392E-02  1.0554E-02  8.5660E-03  6.4838E-03
  3.5000E+01   |   2.4348E-02  2.3181E-02  2.1626E-02  1.9580E-02
  4.0000E+01   |   3.2765E-02  3.3274E-02  3.3650E-02  3.3795E-02
  4.5000E+01   |   3.4099E-02  3.5763E-02  3.7655E-02  3.9828E-02
  5.0000E+01   |   2.9625E-02  3.1360E-02  3.3403E-02  3.5860E-02
  5.5000E+01   |   2.2600E-02  2.3743E-02  2.5052E-02  2.6569E-02
  6.0000E+01   |   1.5672E-02  1.6137E-02  1.6603E-02  1.7048E-02
  6.5000E+01   |   1.0123E-02  1.0119E-02  1.0032E-02  9.8216E-03
  7.0000E+01   |   6.1999E-03  5.9720E-03  5.6534E-03  5.2154E-03
  7.5000E+01   |   3.6474E-03  3.3666E-03  3.0215E-03  2.6027E-03
  8.0000E+01   |   2.0815E-03  1.8329E-03  1.5510E-03  1.2387E-03
  8.5000E+01   |   1.1610E-03  9.7196E-04  7.7211E-04  5.6851E-04
  9.0000E+01   |   6.3660E-04  5.0529E-04  3.7553E-04  2.5382E-04
  9.5000E+01   |   3.4468E-04  2.5884E-04  1.7950E-04  1.1099E-04
  1.0000E+02   |   1.8494E-04  1.3118E-04  8.4708E-05  4.7786E-05

 Lambda
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  Stock Price  |   Time to Maturity (months)
               |   5.0000E+00  4.5000E+00  4.0000E+00  3.5000E+00
-----------------------------------------------------------------
  0.0000E+00   |   0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00
  5.0000E+00   |   8.8525E-17  1.0167E-18  3.8453E-21  2.9781E-24
  1.0000E+01   |   5.5171E-08  6.0702E-09  3.8694E-10  1.1320E-11
  1.5000E+01   |   2.7247E-04  7.7565E-05  1.6224E-05  2.1871E-06
  2.0000E+01   |   2.5496E-02  1.2066E-02  4.7644E-03  1.4538E-03
  2.5000E+01   |   3.6656E-01  2.3400E-01  1.3431E-01  6.6299E-02
  3.0000E+01   |   1.8588E+00  1.4248E+00  1.0279E+00  6.8080E-01
  3.5000E+01   |   4.9710E+00  4.2595E+00  3.5323E+00  2.7983E+00
  4.0000E+01   |   8.7374E+00  7.9857E+00  7.1787E+00  6.3084E+00
  4.5000E+01   |   1.1508E+01  1.0863E+01  1.0167E+01  9.4094E+00
  5.0000E+01   |   1.2344E+01  1.1760E+01  1.1134E+01  1.0459E+01
  5.5000E+01   |   1.1394E+01  1.0773E+01  1.0104E+01  9.3768E+00
  6.0000E+01   |   9.4033E+00  8.7137E+00  7.9693E+00  7.1602E+00
  6.5000E+01   |   7.1285E+00  6.4127E+00  5.6514E+00  4.8412E+00
  7.0000E+01   |   5.0632E+00  4.3894E+00  3.6936E+00  2.9815E+00
  7.5000E+01   |   3.4194E+00  2.8406E+00  2.2661E+00  1.7080E+00
  8.0000E+01   |   2.2203E+00  1.7596E+00  1.3235E+00  9.2488E-01
  8.5000E+01   |   1.3981E+00  1.0534E+00  7.4380E-01  4.7920E-01
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  9.0000E+01   |   8.5941E-01  6.1393E-01  4.0558E-01  2.3986E-01
  9.5000E+01   |   5.1846E-01  3.5040E-01  2.1600E-01  1.1686E-01
  1.0000E+02   |   3.0824E-01  1.9677E-01  1.1294E-01  5.5750E-02

 Rho
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  Stock Price  |   Time to Maturity (months)
               |   5.0000E+00  4.5000E+00  4.0000E+00  3.5000E+00
-----------------------------------------------------------------
  0.0000E+00   |   0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00
  5.0000E+00   |   6.3524E-18  6.5717E-20  2.2112E-22  1.4997E-25
  1.0000E+01   |   5.6040E-09  5.5594E-10  3.1558E-11  8.0937E-13
  1.5000E+01   |   3.6414E-05  9.3595E-06  1.7459E-06  2.0663E-07
  2.0000E+01   |   4.3690E-03  1.8706E-03  6.6008E-04  1.7721E-04
  2.5000E+01   |   7.9884E-02  4.6268E-02  2.3805E-02  1.0371E-02
  3.0000E+01   |   5.1594E-01  3.6026E-01  2.3399E-01  1.3743E-01
  3.5000E+01   |   1.7715E+00  1.3907E+00  1.0448E+00  7.3907E-01
  4.0000E+01   |   4.0509E+00  3.4193E+00  2.8095E+00  2.2269E+00
  4.5000E+01   |   7.0648E+00  6.2263E+00  5.3932E+00  4.5679E+00
  5.0000E+01   |   1.0249E+01  9.2505E+00  8.2458E+00  7.2346E+00
  5.5000E+01   |   1.3108E+01  1.1967E+01  1.0810E+01  9.6342E+00
  6.0000E+01   |   1.5384E+01  1.4101E+01  1.2790E+01  1.1446E+01
  6.5000E+01   |   1.7041E+01  1.5617E+01  1.4153E+01  1.2646E+01
  7.0000E+01   |   1.8167E+01  1.6613E+01  1.5013E+01  1.3363E+01
  7.5000E+01   |   1.8894E+01  1.7231E+01  1.5521E+01  1.3761E+01
  8.0000E+01   |   1.9344E+01  1.7597E+01  1.5806E+01  1.3969E+01
  8.5000E+01   |   1.9615E+01  1.7807E+01  1.5959E+01  1.4072E+01
  9.0000E+01   |   1.9774E+01  1.7924E+01  1.6039E+01  1.4122E+01
  9.5000E+01   |   1.9865E+01  1.7987E+01  1.6080E+01  1.4145E+01
  1.0000E+02   |   1.9917E+01  1.8022E+01  1.6101E+01  1.4156E+01
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